Implementation of special functions

Contents

1 Elliptic integrals and related

1.1 Complete elliptic integral of the first kind

The arithmetic geometric mean $\operatorname{agm}(x,y)$ is defined and calculated as the limit of the iteration:

$$\begin{bmatrix} a_0 \\ g_0 \end{bmatrix} := \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} a_{n+1} \\ g_{n+1} \end{bmatrix} := \begin{bmatrix} \frac{1}{2}(a_n + g_n) \\ \sqrt{a_n g_n} \end{bmatrix}. \tag{1.1}$$

The iteration can be stopped if a_n and g_n are sufficiently close to each other. If this condition fails for some reason, to have a more stable algorithm, a maximum number $n_{\max}$ of iterations should be specified. Numerical experiments show that $n_{\max} = 14$ is enough for 64-bit floating point arithmetic with

$$(x,y) \in [10^{-307}, 10^{308}] \times [10^{-307}, 10^{308}]. \tag{1.2}$$

The complete elliptic integral of the first kind is defined as

$$K(m) := \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{1.3}$$

It is calculated by the arithmetic geometric mean:

$$K(m) = \frac{\pi}{2\operatorname{agm}(1, \sqrt{1-m})}. \tag{1.4}$$

The domain of $K(m)$ is $m < 1$, but (1.3) allows more generally

$$m \in \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}. \tag{1.5}$$

The relation between the arithmetic geometric mean and $K(m)$ holds even for complex numbers, but one has to take care of the branch cut of the square root.

1.2 Complete elliptic integral of the second Kind

The complete elliptic integral of the second kind is defined as

$$E(m) := \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta. \tag{1.6}$$

It is calculated by

$$E(m) = \frac{\pi}{2}\lim_{n\to\infty}\frac{a_n}{x_n}, \tag{1.7}$$

where

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x_n + y_n) \\ \sqrt{x_n y_n} \\ \frac{1}{2}(a_n + b_n) \\ \frac{a_n y_n + b_n x_n}{x_n + y_n} \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \\ a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{1-m} \\ 1 \\ 1-m \end{bmatrix}. \tag{1.8}$$

2 Polynomials and related

2.1 Associated Legendre functions

The associated Legendre functions $P_n^m(x)$ are solutions of the general Legendre equation

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_n^m(x)-2x\frac{\mathrm{d}}{\mathrm{d}x}P_n^m(x)+\Big[(n+1)n-\frac{m^2}{1-x^2}\Big]P_n^m(x)=0. \tag{2.1}$$

In case of $m=n$ one has the recurrence

$$P_0^0(x)=1,\quad P_n^n(x)=-(2n-1)\sqrt{1-x^2}P_{n-1}^{n-1}(x), \tag{2.2}$$

which has the solution

$$P_n^n(x)=(-1)^n(2n-1)!!(1-x^2)^{n/2}. \tag{2.3}$$

By

$$(2n-1)!!=(-2)^n\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)} \tag{2.4}$$

we obtain

$$P_n^n(x)=\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)}(2\sqrt{1-x^2})^n. \tag{2.5}$$

In case of $m=n-1$ one has

$$P_n^{n-1}(x)=(2n-1)xP_{n-1}^{n-1}(x). \tag{2.6}$$

Now we use the recurrence

$$(n-m)P_n^m(x)=(2n-1)xP_{n-1}^m(x)-(n-1+m)P_{n-2}^m(x) \tag{2.7}$$

to get $n\geq m$ down to m. The recurrence will be converted into a bottom up iteration like in the calculation of the Fibonacci sequence. We can remove quadratic complexity by this trick.

This leads us to the following algorithm:

```
function P_n^m(x)
    if n = m
        return sqrt(π)/Γ(1/2−n) (2 sqrt(1−x^2))^n
    else if n−1 = m
        return (2n−1)x P_m^m(x)
    else
        let mut a := P_m^m(x)
        let mut b := P_{m+1}^m(x)
        for k in [m+2..n]
            let h := ((2k−1)xb−(k−1+m)a)/(k−m)
            a := b;  b := h
        end
        return b
    end
end
```

3 Gamma function and related

3.1 Gamma function

The easiest way to compute an approximate value of the gamma function is Stirling's approximation

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x = \sqrt{2\pi}\,x^{x+1/2}\mathrm{e}^{-x}. \tag{3.1}$$

This approximation is also an asymptotic formula. That means, the relative error gets smaller as $x \to \infty$. We can profit from this property if me make x larger by the functional equation

$$\Gamma(x+1) = x\,\Gamma(x). \tag{3.2}$$

Performing the functional equation one time yields again an easy formula, because in this case a simplification is possible:

$$\Gamma(x) \approx \sqrt{2\pi}\,x^{x-1/2}\mathrm{e}^{-x}. \tag{3.3}$$

Formula (3.3) is more precise than (3.1). Iteration of this technique yields

$$\Gamma(x) \approx \sqrt{2\pi}\,\frac{(x+n)^{x+n-1/2}}{\mathrm{e}^{x+n}}\prod_{k=0}^{n-1}\frac{1}{x+k}. \tag{3.4}$$

We can obtain more and more precise values, but convergence as $n \to \infty$ is very slow.

More precise than (3.3) is

$$\Gamma(x) \approx \sqrt{2\pi}\,x^{x-1/2}\exp\Big(-x+\frac{1}{12x}\Big). \tag{3.5}$$

More precise than (3.5) is

$$\Gamma(x) \approx \sqrt{\frac{2\pi}{x}}\bigg(\frac{1}{e}\Big(x+\frac{1}{12x-\frac{1}{10x}}\Big)\bigg)^x, \tag{3.6}$$

found by G. Nemes. More formulas are discussed in [1].

The standard algorithm for 64-bit floating point numbers is Lanczos approximation, see [2].

References

[1] Peter Luschny: *Approximation Formulas for the Factorial Function*.

[2] Glendon Pugh (2004): *An analysis of the Lanczos Gamma approximation*.