

Handbook of Mathematics

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0	0000	0	0
1	0001	1	1
2	0010	2	2
3	0011	3	3
4	0100	4	4
5	0101	5	5
6	0110	6	6
7	0111	7	7
8	1000	8	10
9	1001	9	11
10	1010	A	12
11	1011	B	13
12	1100	C	14
13	1101	D	15
14	1110	E	16
15	1111	F	17

$$\begin{aligned}\sin(-x) &= -\sin x \\ \cos(-x) &= \cos x\end{aligned}$$

$$\begin{aligned}\sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y\end{aligned}$$

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

Polar coordinates

$$\begin{aligned}x &= r \cos \varphi \\ y &= r \sin \varphi \\ \varphi &\in (-\pi, \pi] \\ \det J &= r\end{aligned}$$

Cylindrical coordinates

$$\begin{aligned}x &= r_{xy} \cos \varphi \\ y &= r_{xy} \sin \varphi \\ z &= z \\ \det J &= r_{xy}\end{aligned}$$

Spherical coordinates

$$\begin{aligned}x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \\ z &= r \cos \theta \\ \varphi &\in (-\pi, \pi], \theta \in [0, \pi] \\ \det J &= r^2 \sin \theta\end{aligned}$$

$$\begin{aligned}\theta &= \beta - \pi/2 \\ \beta &\in [-\pi/2, \pi/2] \\ \cos \theta &= \sin \beta \\ \sin \theta &= \cos \beta\end{aligned}$$

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1 Basics and foundations

1.1 Complex numbers

1.1.1 Operations

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}, \quad (1.1)$$

$$\frac{1}{z} = \frac{\bar{z}}{z \bar{z}} = \frac{\bar{z}}{|z|^2}. \quad (1.2)$$

1.1.2 Absolut value

For all $z_1, z_2 \in \mathbb{C}$:

$$|z_1 z_2| = |z_1| |z_2|, \quad (1.3)$$

$$z_2 \neq 0 \implies \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad (1.4)$$

$$z \bar{z} = |z|^2. \quad (1.5)$$

1.1.3 Conjugation

For all $z_1, z_2 \in \mathbb{C}$:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2, \quad (1.6)$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad z_2 \neq 0 \implies \overline{\left(\frac{z_1}{z_2} \right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad (1.7)$$

$$\bar{\bar{z}} = z, \quad |\bar{z}| = |z|, \quad z \bar{z} = |z|^2, \quad (1.8)$$

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}, \quad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}, \quad (1.9)$$

$$\overline{\cos(z)} = \cos(\bar{z}), \quad \overline{\sin(z)} = \sin(\bar{z}), \quad (1.10)$$

$$\overline{\exp(z)} = \exp(\bar{z}). \quad (1.11)$$

1.2 Logic

1.2.1 Propositional logic

1.2.1.1 Boolean algebra

Laws of distributivity:

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \quad (1.12)$$

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C). \quad (1.13)$$

1.2.1.2 Functions in two arguments

There are 16 boolean functions in two arguments.

AB	value
00	a
01	b
10	c
11	d

No.	dcba	Function	Name
0	0000	0	Contradiction
1	0001	$\overline{A \vee B}$	NOR
2	0010	$\overline{B \Rightarrow A}$	
3	0011	\overline{A}	
4	0100	$\overline{A \Rightarrow B}$	
5	0101	\overline{B}	
6	0110	$A \oplus B$	Contravalece
7	0111	$\overline{A \wedge B}$	NAND
8	1000	$A \wedge B$	Conjunction
9	1001	$A \Leftrightarrow B$	Equivalence
10	1010	B	Projection
11	1011	$A \Rightarrow B$	Implication
12	1100	A	Projection
13	1101	$B \Rightarrow A$	Implication
14	1110	$A \vee B$	Disjunction
15	1111	1	Tautology

1.2.2 Predicate logic

1.2.2.1 Basic laws

Negation (De Morgan's laws):

$$\overline{\forall x[P(x)]} \iff \exists x[\overline{P(x)}], \quad (1.14)$$

$$\overline{\exists x[P(x)]} \iff \forall x[\overline{P(x)}]. \quad (1.15)$$

Generalized laws of distributivity:

$$P \vee \forall x[Q(x)] \iff \forall x[P \vee Q(x)], \quad (1.16)$$

$$P \wedge \exists x[Q(x)] \iff \exists x[P \wedge Q(x)]. \quad (1.17)$$

Generalized laws of idempotence:

$$\begin{aligned} \exists x \in M [P] &\iff (M \neq \{\}) \wedge P \\ &\iff \begin{cases} P & \text{if } M \neq \{\}, \\ 0 & \text{if } M = \{\}. \end{cases} \end{aligned} \quad (1.18)$$

$$\begin{aligned} \forall x \in M [P] &\iff (M = \{\}) \vee P \\ &\iff \begin{cases} P & \text{if } M \neq \{\}, \\ 1 & \text{if } M = \{\}. \end{cases} \end{aligned} \quad (1.19)$$

Equivalences:

$$\forall x \forall y [P(x, y)] \iff \forall y \forall x [P(x, y)], \quad (1.20)$$

$$\exists x \exists y [P(x, y)] \iff \exists y \exists x [P(x, y)], \quad (1.21)$$

$$\forall x [P(x) \wedge Q(x)] \iff \forall x [P(x)] \wedge \forall x [Q(x)], \quad (1.22)$$

$$\exists x [P(x) \vee Q(x)] \iff \forall x [P(x)] \vee \forall x [Q(x)], \quad (1.23)$$

$$\forall x [P(x) \Rightarrow Q] \iff \exists x [P(x)] \Rightarrow Q, \quad (1.24)$$

$$\forall x [P \Rightarrow Q(x)] \iff P \Rightarrow \forall x [Q(x)], \quad (1.25)$$

$$\exists x [P(x) \Rightarrow Q(x)] \iff \forall x [P(x)] \Rightarrow \exists x [Q(x)]. \quad (1.26)$$

Implications:

$$\exists x \forall y [P(x, y)] \implies \forall y \exists x [P(x, y)], \quad (1.27)$$

$$\forall x [P(x)] \vee \forall x [Q(x)] \implies \forall x [P(x) \vee Q(x)], \quad (1.28)$$

$$\exists x [P(x) \wedge Q(x)] \implies \exists x [P(x)] \wedge \exists x [Q(x)], \quad (1.29)$$

$$\forall x [P(x) \Rightarrow Q(x)] \implies (\forall x [P(x)] \Rightarrow \forall x [Q(x)]), \quad (1.30)$$

$$\forall x [P(x) \Leftrightarrow Q(x)] \implies (\forall x [P(x)] \Leftrightarrow \forall x [Q(x)]). \quad (1.31)$$

Table 1.1: Operations

Name	Operation	Polar form	Cartesian form
Identity	z	$= re^{i\varphi}$	$= a + bi$
Addition	$z_1 + z_2$		$= (a_1 + a_2) + (b_1 + b_2)i$
Subtraction	$z_1 - z_2$		$= (a_1 - a_2) + (b_1 - b_2)i$
Multiplication	$z_1 z_2$	$= r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$	$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$
Division	$\frac{z_1}{z_2}$	$= \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}$	$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} i$
Reciprocal	$\frac{1}{z}$	$= \frac{1}{r} e^{-i\varphi}$	$= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$
Real part	$\text{Re}(z)$	$= \cos \varphi$	$= a$
Imaginary part	$\text{Im}(z)$	$= \sin \varphi$	$= b$
Conjugation	\bar{z}	$= re^{-i\varphi}$	$= a - bi$
Absolut value	$ z $	$= r$	$= \sqrt{a^2 + b^2}$
Argument	$\arg(z)$	$= \varphi$	$= s(b) \arccos\left(\frac{a}{r}\right)$

$$s(b) := \begin{cases} +1 & \text{if } b \geq 0, \\ -1 & \text{if } b < 0 \end{cases}$$

Table 1.2: Boolean algebra

Disjunction	Conjunction	
$A \vee A \Leftrightarrow A$	$A \wedge A \Leftrightarrow A$	Laws of idempotence
$A \vee 0 \Leftrightarrow A$	$A \wedge 1 \Leftrightarrow A$	Laws of neutrality
$A \vee 1 \Leftrightarrow 1$	$A \wedge 0 = 0$	Laws of annihilation
$A \vee \bar{A} \Leftrightarrow 1$	$A \wedge \bar{A} \Leftrightarrow 0$	Laws of complementation
$A \vee B \Leftrightarrow B \vee A$	$A \wedge B \Leftrightarrow B \wedge A$	Laws of commutativity
$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$	$(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$	Laws of associativity
$A \vee \bar{B} \Leftrightarrow \bar{A} \wedge \bar{B}$	$A \wedge \bar{B} \Leftrightarrow \bar{A} \vee \bar{B}$	De Morgan's laws
$A \vee (A \wedge B) \Leftrightarrow A$	$A \wedge (A \vee B) \Leftrightarrow A$	Laws of absorption

1.2.2.2 Finite sets

Let $M = \{x_1, \dots, x_n\}$. One has:

$$\forall x \in M [P(x)] \Leftrightarrow P(x_1) \wedge \dots \wedge P(x_n), \quad (1.32)$$

$$\exists x \in M [P(x)] \Leftrightarrow P(x_1) \vee \dots \vee P(x_n). \quad (1.33)$$

1.2.2.3 Restricted quantification

$$\forall x \in M [P(x)] :\Leftrightarrow \forall x [x \notin M \vee P(x)] \quad (1.34)$$

$$\Leftrightarrow \forall x [x \in M \Rightarrow P(x)], \quad (1.35)$$

$$\exists x \in M [P(x)] :\Leftrightarrow \exists x [x \in M \wedge P(x)], \quad (1.36)$$

$$\forall x \in M \setminus N [P(x)] \Leftrightarrow \forall x [x \notin N \Rightarrow P(x)]. \quad (1.36)$$

1.2.2.4 Product sets as domains of discourse

$$\forall (x, y) [P(x, y)] \Leftrightarrow \forall x \forall y [P(x, y)], \quad (1.37)$$

$$\exists (x, y) [P(x, y)] \Leftrightarrow \exists x \exists y [P(x, y)]. \quad (1.38)$$

By analogy:

$$\forall (x, y, z) \Leftrightarrow \forall x \forall y \forall z, \quad (1.39)$$

$$\exists (x, y, z) \Leftrightarrow \exists x \exists y \exists z \quad (1.40)$$

etc.

1.2.2.5 Alternative representation

Let $P: G \rightarrow \{0, 1\}$ and $M \subseteq G$. Let $P(M)$ be the image of M under P . One has

$$\forall x \in M [P(x)] \Leftrightarrow P(M) = \{1\} \quad (1.41)$$

$$\Leftrightarrow M \subseteq \{x \in G \mid P(x)\}$$

and

$$\exists x \in M [P(x)] \Leftrightarrow \{1\} \subseteq P(M) \quad (1.42)$$

$$\Leftrightarrow M \cap \{x \in G \mid P(x)\} \neq \emptyset.$$

1.2.2.6 Uniqueness

Quantifier of unique existence:

$$\exists! x [P(x)] \quad (1.43)$$

$$:\Leftrightarrow \exists x [P(x) \wedge \forall y [P(y) \Rightarrow x = y]]$$

$$\Leftrightarrow \exists x [P(x)] \wedge \forall x \forall y [P(x) \wedge P(y) \Rightarrow x = y].$$

1.3 Set theory

1.3.1 Definitions

Subset relation:

$$A \subseteq B :\Leftrightarrow \forall x [x \in A \Rightarrow x \in B]. \quad (1.44)$$

Equality:

$$A = B :\Leftrightarrow \forall x [x \in A \Leftrightarrow x \in B]. \quad (1.45)$$

Union:

$$A \cup B := \{x \mid x \in A \vee x \in B\}. \quad (1.46)$$

Intersection:

$$A \cap B := \{x \mid x \in A \wedge x \in B\}. \quad (1.47)$$

Difference set:

$$A \setminus B := \{x \mid x \in A \wedge x \notin B\}. \quad (1.48)$$

Symmetric difference:

$$A \triangle B := \{x \mid x \in A \oplus x \in B\}. \quad (1.49)$$

1.3.2 Boolean algebra

Laws of distributivity:

$$M \cup (A \cap B) = (M \cup A) \cap (M \cup B), \quad (1.50)$$

$$M \cap (A \cup B) = (M \cap A) \cup (M \cap B). \quad (1.51)$$

1.3.3 Subset relation

Decomposition of equality:

$$A = B \iff A \subseteq B \wedge B \subseteq A. \quad (1.52)$$

Paraphrasing of subset relations:

$$\begin{aligned} A \subseteq B &\iff A \cap B = A \\ &\iff A \cup B = B \\ &\iff A \setminus B = \{\}. \end{aligned} \quad (1.53)$$

Law of contraposition:

$$A \subseteq B = \overline{B} \subseteq \overline{A}. \quad (1.54)$$

1.3.4 Inductive sets

Set theoretical model of the natural numbers:

$$\begin{aligned} 0 &:= \{\}, \quad 1 := \{0\}, \quad 2 := \{0, 1\}, \\ 3 &:= \{0, 1, 2\}, \quad \text{usw.} \end{aligned} \quad (1.55)$$

Successor function:

$$x' := x \cup \{x\}. \quad (1.56)$$

Proof by induction: For a predicate $A(n)$ with $n \in \mathbb{N}$ one has:

$$\begin{aligned} A(n_0) \wedge \forall n \geq n_0 [A(n) \Rightarrow A(n+1)] \\ \implies \forall n \geq n_0 [A(n)]. \end{aligned} \quad (1.57)$$

Table 1.3: Boolean algebra

Union	Intersection	
$A \cup A = A$	$A \cap A = A$	Laws of idempotence
$A \cup \{\} = A$	$A \cap G = A$	Laws of neutrality
$A \cup G = G$	$A \cap \{\} = \{\}$	Laws of annihilation
$A \cup \bar{A} = G$	$A \cap \bar{A} = \{\}$	Laws of complementation
$A \cup B = B \cup A$	$A \cap B = B \cap A$	Laws of commutativity
$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$	Laws of associativity
$\overline{A \cup B} = \bar{A} \cap \bar{B}$	$\overline{A \cap B} = \bar{A} \cup \bar{B}$	De Morgan's laws
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	Laws of absorption

G : Universe

2 Appendix

2.1 Mathematical constants

1. Archimede's constant
 $\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\dots$
2. Euler's number
 $e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\dots$
3. Euler-Mascheroni constant
 $\gamma = 0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\dots$
4. Golden ratio, $(1 + \sqrt{5})/2$
 $\varphi = 1.61803\ 39887\ 49894\ 84820\ 45868\ 34365\dots$
5. First Feigenbaum constant
 $\delta = 4.66920\ 16091\ 02990\ 67185\ 32038\ 20466\dots$
6. Second Feigenbaum constant
 $\alpha = 2.50290\ 78750\ 95892\ 82228\ 39028\ 73218\dots$

2.2 Physical constants

1. Speed of light in vacuum
 $c = 299\ 792\ 458\ \text{m/s}$
2. Electric constant
 $\varepsilon_0 = 8.854\ 187\ 817\ 620\ 39 \times 10^{-12}\ \text{F/m}$
3. Magnetic constant
 $\mu_0 = 4\pi \times 10^{-7}\ \text{H/m}$
4. Elementary charge
 $e = 1.602\ 176\ 6208(98) \times 10^{-19}\ \text{C}$

2.3 Greek alphabet

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ε	Epsilon	R	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Y	υ	Upsilon
I	ι	Iota	Φ	φ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

2.4 Fraktur script

A a	$\mathfrak{A} \mathfrak{a}$	O o	$\mathfrak{O} \mathfrak{o}$
B b	$\mathfrak{B} \mathfrak{b}$	P p	$\mathfrak{P} \mathfrak{p}$
C c	$\mathfrak{C} \mathfrak{c}$	Q q	$\mathfrak{Q} \mathfrak{q}$
D d	$\mathfrak{D} \mathfrak{d}$	R r	$\mathfrak{R} \mathfrak{r}$
E e	$\mathfrak{E} \mathfrak{e}$	S s	$\mathfrak{S} \mathfrak{s}$
F f	$\mathfrak{F} \mathfrak{f}$	T t	$\mathfrak{T} \mathfrak{t}$
G g	$\mathfrak{G} \mathfrak{g}$	U u	$\mathfrak{U} \mathfrak{u}$
H h	$\mathfrak{H} \mathfrak{h}$	V v	$\mathfrak{V} \mathfrak{v}$
I i	$\mathfrak{I} \mathfrak{i}$	W w	$\mathfrak{W} \mathfrak{w}$
J j	$\mathfrak{J} \mathfrak{j}$	X x	$\mathfrak{X} \mathfrak{x}$
K k	$\mathfrak{K} \mathfrak{k}$	Y y	$\mathfrak{Y} \mathfrak{y}$
L l	$\mathfrak{L} \mathfrak{l}$	Z z	$\mathfrak{Z} \mathfrak{z}$
M m	$\mathfrak{M} \mathfrak{m}$		
N n	$\mathfrak{N} \mathfrak{n}$		